The Practical Acoustics of 3D Sound Systems
By Wisam Reid
The Venue
The Venue
Venue Model

FRONT

SIDE

TOP

ISO
Let’s take a stab in the dark!
Proposed Speaker Layout
Proposed Speaker Layout

My first theoretical stab in the dark. The venue would like to leave the 16 Meyer speakers in the ceiling where they are.
Ambisonic Decoder Design

When ideals collide with the Mathematics
A brief introduction to Ambisonics

Ambisonics is a full-sphere surround sound technique: in addition to the horizontal plane, it covers sound sources above and below the listener. Unlike other multi-channel surround formats, its transmission channels do not carry speaker signals.
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Ambisonics is a full-sphere surround sound technique: in addition to the horizontal plane, it covers sound sources above and below the listener. Unlike other multi-channel surround formats, its transmission channels do not carry speaker signals.

An extensible, hierarchical system for representing sound fields

▸ Describes how something should sound, rather than talking about specific speaker signals.
The Ambisonic transmission channels do not carry speaker signals.
A brief introduction to Ambisonics

In mathematics, spherical harmonics are a series of special functions defined on the surface of a sphere used to solve some kinds of differential equations. As Fourier series are a series of functions used to represent functions on a circle, spherical harmonics are a series of functions that are used to represent functions defined on the surface of a sphere. Spherical harmonics are functions defined in terms of spherical coordinates and are organized by angular frequency, as seen in the rows of functions in the illustration on the right.

Spherical harmonics are defined as the angular portion of a set of solutions to Laplace's equation in three dimensions. Represented in a system of spherical coordinates, Laplace's spherical harmonics $Y^m_n$ are a specific set of spherical harmonics that forms an orthogonal system.
“Everything Should Be Made as Simple as Possible, But Not Simpler.”

-- Albert Einstein
The Mathematics of Ambisonics Demystified
Laplace Spherical Harmonics

Definition:

\[ Y_n^m(\theta, \varphi) = \sqrt{\frac{(2n + 1) (n - m)!}{4\pi (n + m)!}} P_n^m(\cos \theta) e^{im\varphi} \]

Laplace Spherical Harmonics

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<table>
<thead>
<tr>
<th>$n$</th>
<th>$Y_n^m(\theta, \phi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$</td>
</tr>
</tbody>
</table>
| 1   | $Y_1^{-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}$  
|     | $Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$  
|     | $Y_1^1(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$ |
| 2   | $Y_2^{-2}(\theta, \phi) = \sqrt{\frac{15}{128\pi}} \sin^2 \theta e^{-2i\phi}$  
|     | $Y_2^{-1}(\theta, \phi) = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{-i\phi}$  
|     | $Y_2^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$  
|     | $Y_2^1(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$  
|     | $Y_2^2(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi}$ |
| 3   | $Y_3^{-3}(\theta, \phi) = \sqrt{\frac{35}{64\pi}} \sin^3 \theta e^{-3i\phi}$  
|     | $Y_3^{-2}(\theta, \phi) = \sqrt{\frac{105}{32\pi}} \cos \theta \sin^2 \theta e^{-2i\phi}$  
|     | $Y_3^{-1}(\theta, \phi) = \sqrt{\frac{21}{64\pi}} (5 \cos^2 \theta - 1) \sin \theta e^{-i\phi}$  
|     | $Y_3^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (5 \cos^3 \theta - 3 \cos \theta)$  
|     | $Y_3^1(\theta, \phi) = -\sqrt{\frac{21}{64\pi}} (5 \cos^2 \theta - 1) \sin \theta e^{i\phi}$  
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|     | $Y_3^3(\theta, \phi) = -\sqrt{\frac{35}{64\pi}} \sin^3 \theta e^{3i\phi}$ |
| 4   | $Y_4^{-4}(\theta, \phi) = \sqrt{\frac{315}{512\pi}} \sin^4 \theta e^{-4i\phi}$  
|     | $Y_4^{-3}(\theta, \phi) = \sqrt{\frac{315}{512\pi}} \cos \theta \sin^3 \theta e^{-3i\phi}$  
|     | $Y_4^{-2}(\theta, \phi) = \sqrt{\frac{45}{128\pi}} (7 \cos^2 \theta - 1) \sin^2 \theta e^{-2i\phi}$  
|     | $Y_4^{-1}(\theta, \phi) = \sqrt{\frac{45}{64\pi}} (7 \cos^3 \theta - 3 \cos \theta) \sin \theta e^{-i\phi}$  
|     | $Y_4^0(\theta, \phi) = \sqrt{\frac{9}{256\pi}} (35 \cos^4 \theta - 30 \cos^2 \theta + 3)$  
|     | $Y_4^1(\theta, \phi) = -\sqrt{\frac{45}{64\pi}} (7 \cos^2 \theta - 3 \cos \theta) \sin \theta e^{i\phi}$  
|     | $Y_4^2(\theta, \phi) = \sqrt{\frac{45}{128\pi}} (7 \cos^2 \theta - 1) \sin^2 \theta e^{2i\phi}$  
|     | $Y_4^3(\theta, \phi) = -\sqrt{\frac{315}{512\pi}} \cos \theta \sin^3 \theta e^{3i\phi}$  
|     | $Y_4^4(\theta, \phi) = \sqrt{\frac{315}{512\pi}} \sin^4 \theta e^{4i\phi}$ |
But, wait …

What in the heck are Legendre polynomials?
The Legendre polynomials form a complete and orthogonal set of basis functions over the line section $x \in [-1,1]$. They are in $L^2([-1,1])$, the space of square-integrable functions on this line section.
Associated Legendre Polynomials

Definition:

\[ P_{n}^{-m} = (-1)^{m} \frac{(n - m)!}{(n + m)!} P_{n}^{m} \]
### Associated Legendre Polynomials

**Definition:**

\[ P_{n}^{-m} = (-1)^{m} \frac{(n - m)!}{(n + m)!} P_{n}^{m} \]

<table>
<thead>
<tr>
<th>n = 0</th>
<th>( P_{0}^{0}(x) = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 1</td>
<td>( P_{1}^{-1}(x) = \frac{1}{2} (1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{1}^{0}(x) = x )</td>
</tr>
<tr>
<td></td>
<td>( P_{1}^{1}(x) = -(1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td>n = 2</td>
<td>( P_{2}^{-2}(x) = \frac{1}{3} (1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{2}^{-1}(x) = \frac{1}{2} x(1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{2}^{0}(x) = \frac{1}{2} (3x^{2} - 1) )</td>
</tr>
<tr>
<td></td>
<td>( P_{2}^{1}(x) = -3x(1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{2}^{2}(x) = 3(1 - x^{2}) )</td>
</tr>
<tr>
<td>n = 3</td>
<td>( P_{3}^{-3}(x) = \frac{1}{48} (1 - x^{2})^{\frac{3}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{3}^{-2}(x) = \frac{1}{4} x(1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{3}^{-1}(x) = \frac{1}{8} (5x^{2} - 1)(1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{3}^{0}(x) = \frac{1}{8} (5x^{3} - 3x) )</td>
</tr>
<tr>
<td></td>
<td>( P_{3}^{1}(x) = -\frac{5}{2} (5x^{2} - 1)(1 - x^{2})^{\frac{1}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{3}^{2}(x) = 15x(1 - x^{2}) )</td>
</tr>
<tr>
<td></td>
<td>( P_{3}^{3}(x) = -15(1 - x^{2})^{\frac{3}{2}} )</td>
</tr>
<tr>
<td>n = 4</td>
<td>( P_{4}^{-4}(x) = \frac{1}{384} (1 - x^{2})^{2} )</td>
</tr>
<tr>
<td></td>
<td>( P_{4}^{-3}(x) = \frac{1}{48} x(1 - x^{2})^{\frac{3}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{4}^{-2}(x) = \frac{1}{48} (7x^{2} - 1)(1 - x^{2}) )</td>
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</tr>
<tr>
<td></td>
<td>( P_{4}^{0}(x) = \frac{1}{8} (35x^{4} - 30x^{2} + 3) )</td>
</tr>
<tr>
<td></td>
<td>( P_{4}^{1}(x) = -\frac{5}{2} (7x^{3} - 3x)(1 - x^{2})^{\frac{1}{2}} )</td>
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</tr>
<tr>
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<td>( P_{4}^{3}(x) = -105x(1 - x^{2})^{\frac{3}{2}} )</td>
</tr>
<tr>
<td></td>
<td>( P_{4}^{4}(x) = 105(1 - x^{2})^{2} )</td>
</tr>
</tbody>
</table>

The complex exponential, widely used in signal processing, forms a complete and orthogonal basis for functions on the circle and is responsible for the behavior of the spherical harmonics as a function of $\varphi$.

The associated Legendre functions for different orders $n$ and the same degree $m$ are orthogonal under integration along $\theta$.

Together, spherical harmonics form an orthogonal and complete system over a sphere surface.
But, wait ...

What are we talking about again?
Balloon Plots:

Isomorphic:

Looking Down X:

Looking Down Y:

Looking Down Z:
Unfortunately ...

Things are not as simple as it may seem
Ambisonics: Caveats in practice

There are many ambisonic conventions: channel weighting and orderings

This chart displays the equations for real valued spherical harmonics implemented in commonly used ambisonic conventions.
Back to Ambisonic Decoder Design!
First of all, ...

What is an **Ambisonic Decoder**?

In Ambisonics, the program format has essentially **vectorized** the soundfield in space and is independent of the reproduction layout.

- The decoder’s task is to create the best perceptual reproduction of given a specific bandwidth, number of speakers, and a configuration thereof.

- We use the term “decoder” to mean the configuration for a decoding engine that does the actual signal processing.

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
The Challenge

We want our decoder to mimic conditions of natural hearing

- Constant amplitude gain for all source directions (P)
- Constant energy gain for all source directions (E)
- At low frequencies, correct reproduced wavefront direction and velocity (rV)
- At high frequencies, maximum concentration of energy in the source direction (rE)
- Matching high and low frequency perceived directions

Getting rE correct is the most difficult aspect and it is also the most important!

Gerzon’s Localization Vector Theory

This Localization Vector Theory ($r_V$ and $r_E$) is motivated by the mechanics of the human auditory localization, uncovered by Psychophysical experimentation.

- At low frequencies $< \sim 800$ Hz auditory perception is dominated by Interaural Time Differences (ITDs)
- At mid range frequencies between $\sim 800$ Hz and $\sim 5$ kHz auditory perception is dominated by Interaural Level Differences (ILDs)

Decoder Design

Decoders for regular polygon and polyhedra loudspeaker arrays are easy to design

- We can build the speaker encoding matrix, $K$, by sampling the spherical harmonics at the speaker directions
- Then use pseudoinverse to find the basic decoding matrix $M$
- $r_E$ guaranteed to point in same direction as $r_V$

Decoders for regular polygon and polyhedra loudspeaker arrays are easy to design

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- Then use pseudoinverse to find the basic decoding matrix $M$
- $r_E$ guaranteed to point in same direction as $r_V$
  - $r_V$, the vector sum of the signals from the loudspeakers
  - $r_E$, the vector sum of the squares of the signals from the loudspeakers.

Gerzon’s Localization Vector Theory

- $r_V$, predicts low-frequency localization almost perfectly.

- $r_E$, predicts mid-frequency localization moderately well.

- Maximizing $r_E$ and getting it to point in the right direction is the crux of the decoder design problem.

- Because this is a non-linear function of speaker position, and virtually all real world arrays are irregular we need to use numerical optimization methods.

Decoder Design

Engineering Tradeoffs

Once we deviate from regular geometry:

- We must trade off localization accuracy for uniform loudness
  - Directions of rE and rV are not the same
- Localization degrades outside the area with a high density of loudspeakers
- Optimization works well for small arrays
- Convergence is slow for large HOA arrays

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Ambisonic Decoder Design

- The AllRAD design technique
- The inversion or mode-matching
- Truncated mode-matching
- Constant energy
- Linear combinations of 2 and 3
- Slepian function basis

There are several mathematical approaches that can be taken.

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Ambisonic Decoder Design

Ambisonic-VBAP approach is called “All Round Ambisonic Decoding” (AllRAD) by Zotter and Frank [2012].

- VBAP always produces the smallest possible angular spread of energy for a given panning direction and speaker array.
- The perceived size of a virtual source changes depending on direction.

**This achieved by:**
- The number of virtual speakers is made much larger than the number of real speakers.
- Ghost speakers are inserted to fill in large gaps in the real loudspeaker array in order to keep the triangular tessellation of virtual speaker directions as regular as possible.

At odds with the Ambisonic approach, which tries to keep the perceived size of a virtual source constant regardless of source direction.

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Ambisonic Decoder Design

- One solution is to set all of the singular values to 1 when computing the pseudoinverse [Pomberger and Zotter 2012]. The resulting decoder has constant energy (loudness) in all directions, at the expense of increased directional errors.

- Another solution is to use a truncated SVD when computing the pseudoinverse. This simply discards the poorly sampled spherical harmonics.

- Setting an upper limit on loudness variations, at the expense of increased directional errors.

- The ADT toolbox can produce decoders that are a linear combinations of these, trading off uniform loudness and directional accuracy.

The inversion:

- Mode-matching
- Truncated mode-matching
- Constant energy

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Ambisonic Decoder Design

- Spherical Slepian functions (SSF) are linear combinations of spherical harmonics that produce new basis functions that are approximately zero outside the chosen region of the sphere, but also remain orthogonal within the region of interest.

- This is called, “Energy-Preserving Ambisonic Decoding” (EPAD)

- This is implemented, in part, using a Gramian matrix, the inner products of the real spherical harmonics,

- This method creates basis functions that have a clearer relationship with source directions, which is not possible for the spherical harmonics above first order.

Spherical Slepian Function Decoding

There are several mathematical approaches that can be taken.

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Ambisonic Decoder Design

Spherical Slepian Function Decoding
“Energy-Preserving Ambisonic Decoding” (EPAD)

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Okay so wait ...

So, what does this look like in practice?
What does this look like in practice?

Let’s do it ideally (as possible) first!

CCRMA Listening room speaker locations

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Let’s do it ideally (as possible) first!

CCRMA Listening Room ALLRAD

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Let’s do it ideally (as possible) first!

CCRMA Listening Room Slepian rV grid

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Let’s do it ideally (as possible) first!

CCRMA Listening Room Slepian rE grid

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Let’s do it ideally (as possible) first!

CCRMA Listening Room Slepian rV performance

A. Heller and E. M. Benjamin, "The Ambisonic Decoder Toolbox" Linux Audio Conference 2014
Let’s do it ideally (as possible) first!

CCRMA Listening Room Slepian rE performance

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Let’s Experiment!

Remember our proposed speaker layout?
Our design and layout is under many practical constraints.

Is this feasible?
Results

The Village
ALLRAD Results

Speaker Layout
ALLRAD Results

The Village rE grid
ALLRAD Results

The Village \( rV \) grid
ALLRAD Results

The Village $r_V$ versus $r_E$ direction error
ALLRAD Results

The Village rE performance
ALLRAD Results

The Village rV performance
Results

CCRMA Stage
CCRMA Stage

Speaker Layout
CCRMA Stage

CCRMA Stage rE grid

amb 3H3V Sleipian11
CCRMA Stage

CCRMA Stage rV grid

amb 3H3V Sleplan11
CCRMA Stage

CCRMA Stage $r_V$ versus $r_E$ direction error
CCRMA Stage

CCRMA Stage rE performance
CCRMA Stage

CCRMA Stage rV performance
Turenas Up Mix
Turenas Up Mix

Top View

Side View
Conclusions

ALLRAD Performed Best
The mode-matching techniques are at a loss in this arrangement due to extremely poor spherical sampling in speaker placement. SSF results in extreme warping in the rV vector direction.

The speaker layout constraints need to be revisited
The top 16 speakers being constrained to the ceiling is a gross misuse of speakers in extremely suboptimal locations ~ 7 meters above the audience.

More creative use of the speakers given constraints needs to be explored
New ideas including rotating, staggering and dilating the speaker rings separately may help with warping effects.

Other Challenges
The size of the venue causes spatial aliasing with smaller numbers of speakers. Exploring how performance increases as a function of the number of speakers needs to be analyzed. When will increasing ambisonic decoding order and the number of speakers used saturate performance increases?
Future Work

While the groundwork has been laid there is a lot more work ahead to complete the acoustic aspects of this project.

- Speaker Decorrelation
- Acoustic measurements and compensations
- Decoder optimizations
- Meyer constellation installation
- Validate with ITD and ILD measurements
- Validate through VR psychophysics experiments
References


Thank You
Definitions

A position \( r = (r, \theta, \varphi) \) represented in spherical coordinates can be related to the same position represented in Cartesian coordinates \( x = (x, y, z) \) using [1]

\[
\begin{align*}
x &= r \sin \theta \cos \varphi \\
y &= r \sin \theta \sin \varphi \\
z &= r \cos \theta
\end{align*}
\]

Functions on the unit sphere are presented as a weighted sum of a set of basis functions, also forming the Fourier basis for functions on the sphere.

The Laplace spherical harmonics are generally defined as [1]:

\[
Y_n^m(\theta, \varphi) = \sqrt{\frac{(2n + 1)}{4\pi}} \frac{(n - m)!}{(n + m)!} P_n^m(\cos \theta) e^{im\varphi}
\]

(4)

\( P_n^m(\cdot) \) are the associated Legendre functions denoting the function order.

Several different associated Legendre polynomial normalizations are in common use for the Laplace spherical harmonic functions. The standard convention is defined as [1]:

\[
P_n^{-m} = (-1)^m \frac{(n - m)!}{(n + m)!} P_n^m
\]

which is the natural normalization given by Rodrigues’ formula.

* We will discuss how different weightings are used by different ambisonic conventions.

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*Department of Music, Stanford University, Stanford, CA, 94305 USA; Student ID#: 06043595 e-mail: wisam@ccrma.stanford.edu.
Let’s compute some harmonics

Since the size of the space requires a large number of speakers (48 proposed), our decoder we will need vertical and horizontal ambisonic orders up to order 5, let’s see what they look like.

For Order 0 \((n = 0):\)

\[
Y_0^0(\theta, \varphi) = \frac{1}{2} \sqrt{\frac{1}{\pi}}
\]

For Order 1 \((n = 1):\)

\[
\begin{align*}
Y_1^{-1}(\theta, \varphi) &= \frac{1}{2} \sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin \theta = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \cdot \frac{(x - iy)}{r} \\
Y_1^0(\theta, \varphi) &= \frac{1}{2} \sqrt{\frac{3}{2\pi}} \cdot \cos \theta = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \cdot \frac{z}{r} \\
Y_1^1(\theta, \varphi) &= \frac{1}{2} \sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin \theta = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \cdot \frac{(x + iy)}{r}
\end{align*}
\]

For Order 2 \((n = 2):\)

\[
\begin{align*}
Y_2^{-2}(\theta, \varphi) &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2 \theta = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \cdot \frac{(x - iy)^2}{r^2} \\
Y_2^{-1}(\theta, \varphi) &= \frac{1}{2} \sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin \theta \cdot \cos \theta = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \cdot \frac{(x - iy)z}{r^2} \\
Y_2^0(\theta, \varphi) &= \frac{1}{4} \sqrt{\frac{5}{\pi}} \cdot (3 \cos^2 \theta - 1) = \frac{1}{4} \sqrt{\frac{5}{\pi}} \cdot \frac{(2z^2 - x^2 - y^2)}{r^2} \\
Y_2^1(\theta, \varphi) &= \frac{-1}{2} \sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin \theta \cdot \cos \theta = \frac{-1}{2} \sqrt{\frac{15}{2\pi}} \cdot \frac{(x + iy)z}{r^2} \\
Y_2^2(\theta, \varphi) &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2 \theta = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \cdot \frac{(x + iy)^2}{r^2}
\end{align*}
\]
For Order 3 \((n = 3)\):

\[
Y_3^{-3}(\theta, \varphi) = \frac{1}{8} \sqrt{\frac{35}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3 \theta = \frac{1}{8} \sqrt{\frac{35}{\pi}} \cdot \frac{(x - iy)^3}{r^3}
\]

\[
Y_3^{-2}(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2 \theta \cdot \cos \theta = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \cdot \frac{(x - iy)^2 z}{r^3}
\]

\[
Y_3^{-1}(\theta, \varphi) = \frac{1}{8} \sqrt{\frac{21}{\pi}} \cdot e^{-i\varphi} \cdot \sin \theta \cdot (5 \cos^2 \theta - 1) = \frac{1}{8} \sqrt{\frac{21}{\pi}} \cdot \frac{(x - iy)(4z^2 - x^2 - y^2)}{r^3}
\]

\[
Y_3^{0}(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{7}{\pi}} \cdot (5 \cos^3 \theta - 3 \cos \theta) = \frac{1}{4} \sqrt{\frac{7}{\pi}} \cdot \frac{z(2z^2 - 3x^2 - 3y^2)}{r^3}
\]

\[
Y_3^{1}(\theta, \varphi) = -\frac{1}{8} \sqrt{\frac{21}{\pi}} \cdot e^{i\varphi} \cdot \sin \theta \cdot (5 \cos^2 \theta - 1) = -\frac{1}{8} \sqrt{\frac{21}{\pi}} \cdot \frac{(x + iy)(4z^2 - x^2 - y^2)}{r^3}
\]

\[
Y_3^{2}(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2 \theta \cdot \cos \theta = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \cdot \frac{(x + iy)^2 z}{r^3}
\]

\[
Y_3^{3}(\theta, \varphi) = -\frac{1}{8} \sqrt{\frac{35}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3 \theta = -\frac{1}{8} \sqrt{\frac{35}{\pi}} \cdot \frac{(x + iy)^3}{r^3}
\]

For Order 4 \((n = 4)\):

\[
Y_4^{-4}(\theta, \varphi) = \frac{3}{16} \sqrt{\frac{35}{2\pi}} \cdot e^{-4i\varphi} \cdot \sin^4 \theta = \frac{3}{16} \sqrt{\frac{35}{2\pi}} \cdot \frac{(x - iy)^4}{r^4}
\]

\[
Y_4^{-3}(\theta, \varphi) = \frac{3}{8} \sqrt{\frac{35}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3 \theta \cdot \cos \theta = \frac{3}{8} \sqrt{\frac{35}{\pi}} \cdot \frac{(x - iy)^3 z}{r^4}
\]

\[
Y_4^{-2}(\theta, \varphi) = \frac{3}{8} \sqrt{\frac{5}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2 \theta \cdot (7 \cos^2 \theta - 1) = \frac{3}{8} \sqrt{\frac{5}{2\pi}} \cdot \frac{(x - iy)^2 (7z^2 - r^2)}{r^4}
\]

\[
Y_4^{-1}(\theta, \varphi) = \frac{3}{8} \sqrt{\frac{5}{\pi}} \cdot e^{-i\varphi} \cdot \sin \theta \cdot (7 \cos^3 \theta - 3 \cos \theta) = \frac{3}{8} \sqrt{\frac{5}{\pi}} \cdot \frac{(x - iy) \cdot z \cdot (7z^2 - 3r^2)}{r^4}
\]

\[
Y_4^{0}(\theta, \varphi) = \frac{3}{16} \sqrt{\frac{1}{\pi}} \cdot (35 \cos^4 \theta - 30 \cos^2 \theta + 3) = \frac{3}{16} \sqrt{\frac{1}{\pi}} \cdot \frac{(35z^4 - 30z^2 r^2 + 3r^4)}{r^4}
\]

\[
Y_4^{1}(\theta, \varphi) = -\frac{3}{8} \sqrt{\frac{5}{\pi}} \cdot e^{i\varphi} \cdot \sin \theta \cdot (7 \cos^3 \theta - 3 \cos \theta) = -\frac{3}{8} \sqrt{\frac{5}{\pi}} \cdot \frac{(x + iy) \cdot z \cdot (7z^2 - 3r^2)}{r^4}
\]

\[
Y_4^{2}(\theta, \varphi) = \frac{3}{8} \sqrt{\frac{5}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2 \theta \cdot (7 \cos^2 \theta - 1) = \frac{3}{8} \sqrt{\frac{5}{2\pi}} \cdot \frac{(x + iy)^2 (7z^2 - r^2)}{r^4}
\]

\[
Y_4^{3}(\theta, \varphi) = -\frac{3}{8} \sqrt{\frac{35}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3 \theta \cdot \cos \theta = -\frac{3}{8} \sqrt{\frac{35}{\pi}} \cdot \frac{(x + iy)^3 z}{r^4}
\]

\[
Y_4^{4}(\theta, \varphi) = \frac{3}{16} \sqrt{\frac{35}{2\pi}} \cdot e^{4i\varphi} \cdot \sin^4 \theta = \frac{3}{16} \sqrt{\frac{35}{2\pi}} \cdot \frac{(x + iy)^4}{r^4}
\]
For Order 5 \((n = 5)\):

\[
Y_5^{-5}(\theta, \varphi) = \frac{3}{32} \sqrt{\frac{77}{\pi}} \cdot e^{-5i\varphi} \cdot \sin^5 \theta \\
Y_5^{-4}(\theta, \varphi) = \frac{3}{16} \sqrt{\frac{385}{2\pi}} \cdot e^{-4i\varphi} \cdot \sin^4 \theta \cdot \cos \theta \\
Y_5^{-3}(\theta, \varphi) = \frac{1}{32} \sqrt{\frac{385}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3 \theta \cdot (9\cos^2 \theta - 1) \\
Y_5^{-2}(\theta, \varphi) = \frac{1}{8} \sqrt{\frac{1155}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2 \theta \cdot (3\cos^3 \theta - \cos \theta) \\
Y_5^{-1}(\theta, \varphi) = \frac{1}{16} \sqrt{\frac{165}{2\pi}} \cdot e^{-i\varphi} \cdot \sin \theta \cdot (21\cos^4 \theta - 14\cos^2 \theta + 1) \\
Y_5^0(\theta, \varphi) = \frac{1}{16} \sqrt{\frac{11}{\pi}} \cdot (63\cos^5 \theta - 70\cos^3 \theta + 15\cos \theta) \\
Y_5^1(\theta, \varphi) = \frac{1}{16} \sqrt{\frac{165}{2\pi}} \cdot e^{i\varphi} \cdot \sin \theta \cdot (21\cos^4 \theta - 14\cos^2 \theta + 1) \\
Y_5^2(\theta, \varphi) = \frac{1}{8} \sqrt{\frac{1155}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2 \theta \cdot (3\cos^3 \theta - \cos \theta) \\
Y_5^3(\theta, \varphi) = \frac{1}{32} \sqrt{\frac{385}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3 \theta \cdot (9\cos^2 \theta - 1) \\
Y_5^4(\theta, \varphi) = \frac{3}{16} \sqrt{\frac{385}{2\pi}} \cdot \sin^4 \theta \cdot \cos \theta \\
Y_5^5(\theta, \varphi) = \frac{-3}{32} \sqrt{\frac{77}{\pi}} \cdot e^{5i\varphi} \cdot \sin^5 \theta
\]

An interesting side note:

Real spherical harmonics correspond to atomic orbital symbols \((s, p, d, f, g)\).

References